1 積分技巧雜談

對於

$$\int_{-3}^{3} \sqrt{9-x^2}\,\mathrm{d}x$$

通常會想到做三角代換。不過積分就是在求面積，而被積函數 $y = \sqrt{3^2 - x^2}$ 其實就是上半圓，其半徑為 3，所以直接使用圓面積公式便可得到

$$\int_{-3}^{3} \sqrt{9-x^2}\,\mathrm{d}x = \frac{1}{2}\cdot 3^2\pi$$

除非考試題目直接指明使用三角代換，否則遇到這個直接使用圓面積就出來了。

有了這經驗後，我們來看

$$\int_{-2}^{2} (x+3)\sqrt{4-x^2}\,\mathrm{d}x$$

先乘開寫成

$$\begin{aligned}&\int_{-2}^{2} x\sqrt{4-x^2} + 3\sqrt{4-x^2}\,\mathrm{d}x\\ =&\int_{-2}^{2} x\sqrt{4-x^2}\,\mathrm{d}x + 3\int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x\end{aligned}$$

拆開以後，注意第一項是奇函數的積分，結果為 0；第二項是上半圓的面積再三倍，所以答案就是

$$\int_{-2}^{2} (x+3)\sqrt{4-x^2}\,\mathrm{d}x = 0 + 3\cdot\frac{1}{2}\cdot 2^2\pi = 6\pi$$

其實應該說一開始就注意到有一塊奇函數的部分，才會想要一開始乘開。現在將這種對稱性的考慮昇華到暴力層次，請看下題。

例題 1.1

$$\int_{-2}^{2} x\ln\left(1+e^x\right)\mathrm{d}x$$

解

積分範圍的對稱讓我們想到是否可以利用奇偶性。首先分析若 $f(x) = x\ln\left(1+e^x\right)$，則 $f(-x) = -x\ln\left(1+e^{-x}\right) = -x\ln\left(\frac{e^x+1}{e^x}\right) = -x\ln\left(1+e^x\right) + x\ln e^x = -x\ln\left(1+e^x\right) + x^2$。差了一點點，實在是壯士扼腕！百般不服氣之下，決定霸王硬上弓，拆成

$$\begin{aligned}&\int_{-2}^{2} x\ln\left(1+e^x\right) - \frac{x^2}{2}\,\mathrm{d}x + \int_{-2}^{2}\frac{x^2}{2}\,\mathrm{d}x\\ =&\,0 + \int_{0}^{2} x^2\,\mathrm{d}x\end{aligned}$$

其中 $g(x) = x\ln\left(1+e^x\right) - \frac{x^2}{2}, g(-x) = -x\ln\left(1+e^x\right) + x^2 - \frac{x^2}{2} = -g(x)$，故 $g(x)$ 為奇函數。

接下來看一個可以作巧妙互消的積分

$$\int \frac{xe^x}{(x+1)^2}\,\mathrm{d}x$$

先作個 $u = x + 1$ 代換讓分母好看一點

$$= \int \frac{(u-1)e^{(u-1)}}{u^2}\,\mathrm{d}u = \frac{1}{e}\cdot\left(\int \frac{e^u}{u}\,\mathrm{d}u - \int \frac{e^u}{u^2}\,\mathrm{d}u\right) \quad \boxed{\text{對第一個做分部積分}}$$

$$= \frac{1}{e}\cdot\left(\frac{e^u}{u} + \cancel{\int \frac{e^u}{u^2}\,\mathrm{d}u} - \cancel{\int \frac{e^u}{u^2}\,\mathrm{d}u}\right)$$

$$= \frac{e^{u-1}}{u} + C = \frac{e^x}{x+1} + C$$

當遇到這種帶有 e^x 的積分

$$\int \frac{e^x}{e^x+1}\,\mathrm{d}x$$

只要你在學習變數代換技巧時，有培養一定的靈敏度，便知道只要做 $u = e^x$ 的代換即可作出。然而如果我將分子改成 1，變成

$$\int \frac{dx}{e^x+1}$$

那麼就上下同乘以 e^{-x} 來強迫湊微分，得到

$$\int \frac{e^{-x}dx}{1+e^{-x}}$$

接著再設 $u = e^{-x}$ 即可。

例題 1.2

$$\int_{-1}^{1} \frac{\mathrm{d}x}{(e^x+1)(1+x^2)}$$

解

$$\int_{-1}^{0} \frac{\mathrm{d}x}{(e^x+1)(1+x^2)} + \int_{0}^{1} \frac{\mathrm{d}x}{(e^x+1)(1+x^2)}$$

$$= -\int_{1}^{0} \frac{\mathrm{d}u}{(e^{-u}+1)(1+u^2)} + \int_{0}^{1} \frac{\mathrm{d}x}{(e^x+1)(1+x^2)}$$

$$= \int_{0}^{1} \frac{e^u\,\mathrm{d}u}{(1+e^u)(1+u^2)} + \int_{0}^{1} \frac{\mathrm{d}x}{(e^x+1)(1+x^2)}$$

$$= \int_{0}^{1} \frac{\mathrm{d}u}{1+u^2} = \frac{\pi}{4}$$

例題 1.3

$$\int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}}$$

解 1

看到那個 $1-x^2$，比較直覺地會想到三角代換 $x=\sin(t)$

$$\int \frac{\cos(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \int \frac{\cos(t)\big(\sin(t)+\cos(t)\big)}{\big(\sin(t)+\cos(t)\big)^2}\,\mathrm{d}t = \frac{1}{2}\int \frac{\sin(2t)+\big(1+\cos(2t)\big)}{1+\sin(2t)}\,\mathrm{d}t$$

$$= \frac{1}{2}\int \mathrm{d}t + \frac{1}{2}\int \frac{\cos(2t)}{1+\sin(2t)}\,\mathrm{d}t = \boxed{\frac{1}{2}t + \frac{1}{4}\ln\left|1+\sin(2t)\right| + C}$$

$$= \boxed{\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}\ln\left|x+\sqrt{1-x^2}\right| + C}$$

解 2

$$\int \frac{\cos(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \int \frac{\cos(t)\big(\cos(t)-\sin(t)\big)}{\cos^2(t)-\sin^2(t)}\,\mathrm{d}t = \frac{1}{2}\int \frac{\big(1+\cos(2t)\big)-\sin(2t)}{\cos(2t)}\,\mathrm{d}t$$

$$= \frac{1}{2}\int \big(\sec(2t)+1-\tan(2t)\big)\,\mathrm{d}t = \frac{1}{4}\Big(\ln\left|\sec(2t)+\tan(2t)\right| + 2t - \ln\left|\sec(2t)\right|\Big) + C$$

$$= \boxed{\frac{1}{2}t + \frac{1}{4}\ln\left|1+\sin(2t)\right| + C}$$

解 3

$$\int \frac{\cos(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \int \frac{\cos\left[(t+\frac{\pi}{4})-\frac{\pi}{4}\right]}{\sqrt{2}\sin(t+\frac{\pi}{4})}\,\mathrm{d}t = \int \frac{\frac{1}{\sqrt{2}}\cos(t+\frac{\pi}{4})+\frac{1}{\sqrt{2}}\sin(t+\frac{\pi}{4})}{\sqrt{2}\sin(t+\frac{\pi}{4})}\,\mathrm{d}t$$

$$= \frac{1}{2}\int \Big(\cot(t+\frac{\pi}{4})+1\Big)\,\mathrm{d}t = \frac{1}{2}t + \frac{1}{2}\ln\left|\sin\Big(t+\frac{\pi}{4}\Big)\right| + C$$

$$= \frac{1}{2}t + \frac{1}{2}\ln\left|\frac{1}{\sqrt{2}}\sin(t)+\frac{1}{\sqrt{2}}\cos(t)\right| + C = \frac{1}{2}t + \frac{1}{2}\ln\left|\sin(t)+\cos(t)\right| - \frac{1}{2}\ln\frac{1}{\sqrt{2}} + C$$

$$= \boxed{\frac{1}{2}t + \frac{1}{4}\ln\left|1+\sin(2t)\right| + C}$$

解 4

$$\int \frac{\cos(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \int \frac{1}{1+\tan(t)}\,\mathrm{d}t = \int \frac{\sec^2(t)}{\big(1+\tan(t)\big)\big(1+\tan^2(t)\big)}\,\mathrm{d}t$$

$$= \frac{1}{2}\int \left(\frac{1}{1+\tan(t)} + \frac{1-\tan(t)}{1+\tan^2(t)}\right)\sec^2(t)\,\mathrm{d}t$$

例題 1.3

$$
\begin{aligned}
&= \frac{1}{2}\int\left(\frac{1}{1+\tan(t)}+\frac{1}{1+\tan^2(t)}-\frac{\tan(t)}{1+\tan^2(t)}\right)\sec^2(t)\,\mathrm{d}t \\
&= \frac{1}{2}\int \mathrm{d}t+\frac{1}{2}\int\left(\frac{1}{1+\tan(t)}-\frac{\tan(t)}{1+\tan^2(t)}\right)\sec^2(t)\,\mathrm{d}t \\
&= \frac{1}{2}t+\frac{1}{2}\int\frac{\mathrm{d}u}{1+u}-\frac{1}{2}\int\frac{u\,\mathrm{d}u}{1+u^2} = \frac{1}{2}t+\frac{1}{2}\ln|1+u|-\frac{1}{4}\ln\left|1+u^2\right|+C \\
&= \frac{1}{2}t+\frac{1}{2}\ln\left|1+\tan(t)\right|-\frac{1}{4}\ln\left|1+\tan^2(t)\right|+C = \frac{1}{2}t+\frac{1}{4}\ln\left|\frac{\left(1+\tan(t)\right)^2}{\sec^2(t)}\right|+C \\
&= \frac{1}{2}t+\frac{1}{4}\ln\left|1+\frac{2\tan(t)}{\sec^2(t)}\right|+C = \boxed{\frac{1}{2}t+\frac{1}{4}\ln\left|1+\sin(2t)\right|+C}
\end{aligned}
$$

解 5

$$
\begin{aligned}
&\int\frac{\cos(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \frac{1}{2}\int\frac{\sin(t)+\cos(t)+\cos(t)-\sin(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t \\
&= \frac{1}{2}\int \mathrm{d}t+\frac{1}{2}\int\frac{\cos(t)-\sin(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \frac{1}{2}\int \mathrm{d}t+\frac{1}{2}\ln\left|\sin(t)+\cos(t)\right|+C \\
&= \boxed{\frac{1}{2}t+\frac{1}{4}\ln\left|1+\sin(2t)\right|+C}
\end{aligned}
$$

解 6

三角代換後可用 Weierstrass 代換 $u=\tan(\frac{t}{2}), \mathrm{d}t=\frac{2\,\mathrm{d}u}{1+u^2}, \sin(t)=\frac{2u}{1+u^2}, \cos(t)=\frac{1-u^2}{1+u^2}$

$$
\begin{aligned}
&\int\frac{\cos(t)}{\sin(t)+\cos(t)}\,\mathrm{d}t = \int\frac{\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\frac{2\,\mathrm{d}u}{1+u^2} \\
&= \int\frac{2(1-u^2)}{(1+2u-u^2)(1+u^2)}\,\mathrm{d}u = \int\left(\frac{1-u}{1+2u-u^2}+\frac{1-u}{1+u^2}\right)\mathrm{d}u \\
&= \boxed{\frac{1}{2}t+\frac{1}{4}\ln\left|1+\sin(2t)\right|+C}
\end{aligned}
$$

解 7

直接嘗試 $u=x+\sqrt{1-x^2}, \mathrm{d}u=\left(1-\frac{x}{\sqrt{1-x^2}}\right)\mathrm{d}x$，為了處理多出的 $\frac{x}{\sqrt{1-x^2}}$，拆成

$$
\begin{aligned}
&\int\frac{\mathrm{d}x}{x+\sqrt{1-x^2}} \\
&= \int\frac{1-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}}{x+\sqrt{1-x^2}}\,\mathrm{d}x = \int\frac{\mathrm{d}u}{u}+\int\frac{\frac{x}{\sqrt{1-x^2}}}{x+\sqrt{1-x^2}}\,\mathrm{d}x
\end{aligned}
$$

例題 1.3

$$= \ln|u| + \int \frac{\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x}{x+\sqrt{1-x^2}} = \ln|u| + \int \frac{\mathrm{d}x}{\left(x+\sqrt{1-x^2}\right)\sqrt{1-x^2}}$$

$$= \ln|u| + \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} - \int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}}$$

$$= \ln|u| + \sin^{-1}(x) - \int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}} + C$$

出現原積分，故由自套法得

$$\int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}} = \boxed{\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}\ln\left|x+\sqrt{1-x^2}\right| + C}$$

解 8

$$\int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}} = \int \frac{\sqrt{1-x^2}}{\left(x+\sqrt{1-x^2}\right)\sqrt{1-x^2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{\left(x+\sqrt{1-x^2}\right) - \left(x-\sqrt{1-x^2}\right)}{\left(x+\sqrt{1-x^2}\right)\sqrt{1-x^2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x + \frac{1}{2}\int \frac{1-\frac{x}{\sqrt{1-x^2}}}{x+\sqrt{1-x^2}}\,\mathrm{d}x$$

$$= \boxed{\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}\ln\left|x+\sqrt{1-x^2}\right| + C}$$

以下介紹一些更深入的技巧，這些並不是一般微積分課程所必須知道的，所以一般讀者可以跳過。

定理 1

若 $f(x)$ 在 $[a,b]$ 上連續，則

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x = \frac{1}{2}\int_a^b \left(f(x)+f(a+b-x)\right)\mathrm{d}x$$

證 設 $x = a+b-t, \mathrm{d}x = -\mathrm{d}t$，則

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(a+b-t)\,\mathrm{d}t = \int_a^b f(a+b-x)\,\mathrm{d}x$$

$$= \int_a^b \left(f(x)+f(a+b-x)\right)\mathrm{d}x$$

■

下題便是一個應用。

例題 1.4

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x$$

解 1

常見的做法是先設 $x = \tan(t), \mathrm{d}x = \sec^2(t)\,\mathrm{d}t$，則

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \ln\Big(1+\tan(t)\Big)\,\mathrm{d}t \\
=&\int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos(t)+\sin(t)}{\cos(t)}\right)\mathrm{d}t \\
=&\int_0^{\frac{\pi}{4}} \ln\Big(\cos(t)+\sin(t)\Big)\,\mathrm{d}t - \int_0^{\frac{\pi}{4}} \ln\Big(\cos(t)\Big)\,\mathrm{d}t \\
=&\int_0^{\frac{\pi}{4}} \ln\Big(\sqrt{2}\cos\Big(\frac{\pi}{4}-t\Big)\Big)\,\mathrm{d}t - \int_0^{\frac{\pi}{4}} \ln\Big(\cos(t)\Big)\,\mathrm{d}t \\
=&\int_0^{\frac{\pi}{4}} \ln\sqrt{2}\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln\Big(\cos\Big(\frac{\pi}{4}-t\Big)\Big)\,\mathrm{d}t - \int_0^{\frac{\pi}{4}} \ln\Big(\cos(t)\Big)\,\mathrm{d}t \\
=&\frac{\pi}{4}\ln\sqrt{2} + \int_0^{\frac{\pi}{4}} \ln\Big(\cos(u)\Big)\,\mathrm{d}u - \int_0^{\frac{\pi}{4}} \ln\Big(\cos(t)\Big)\,\mathrm{d}t \\
=&\frac{\pi}{8}\ln(2)
\end{aligned}$$

解 2

先設 $x = \tan(t), \mathrm{d}x = \sec^2(t)\,\mathrm{d}t$，則

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \ln\Big(1+\tan(t)\Big)\,\mathrm{d}t \\
=&\frac{1}{2}\int_0^{\frac{\pi}{4}} \Big(\ln\Big(1+\tan(t)\Big) + \ln(1+\tan\Big(\frac{\pi}{4}-t\Big)\Big)\,\mathrm{d}t \\
=&\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\Big(\Big(1+\tan(t)\Big)\Big[1+\tan\Big(\frac{\pi}{4}-t\Big)\Big]\Big)\,\mathrm{d}t \\
=&\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,\mathrm{d}t \\
=&\frac{\pi}{8}\ln(2)
\end{aligned}$$

另外一個變形是：

定理 2

若 $f(x)$ 在 $[a,b]$ 上連續，則

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^{\frac{a+b}{2}} \big(f(x)+f(a+b-x)\big)\,\mathrm{d}x = \int_{\frac{a+b}{2}}^b \big(f(x)+f(a+b-x)\big)\,\mathrm{d}x$$

證

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_a^{\frac{a+b}{2}} f(x)\,\mathrm{d}x + \int_{\frac{a+b}{2}}^b f(x)\,\mathrm{d}x \\ &= \int_a^{\frac{a+b}{2}} f(x)\,\mathrm{d}x - \int_{\frac{a+b}{2}}^a f(a+b-t)\,\mathrm{d}t = \int_a^{\frac{a+b}{2}} \big(f(x)+f(a+b-x)\big)\,\mathrm{d}x\end{aligned}$$

■

例題 1.5

$$\int_{-\pi}^{\pi} \frac{x\sin^3(x)\tan^{-1}(e^x)}{1+\cos^2(x)}\,\mathrm{d}x$$

解

此題 $a=-\pi, b=\pi$，故有 $\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_0^{\pi}\big(f(x)+f(-x)\big)\,\mathrm{d}x$。所以

$$\begin{aligned}&\int_{-\pi}^{\pi} \frac{x\sin^3(x)\tan^{-1}(e^x)}{1+\cos^2(x)}\,\mathrm{d}x \\ &= \int_0^{\pi}\left(\frac{x\sin^3(x)\tan^{-1}(e^x)}{1+\cos^2(x)} + \frac{x\sin^3(x)\tan^{-1}(e^{-x})}{1+\cos^2(x)}\right)\end{aligned}$$

由於 $\tan^{-1}(u)+\tan^{-1}(\frac{1}{u}) = \frac{\pi}{2}$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{x\sin^3(x)}{1+\cos^2(x)}$$

又由定理 1

$$\begin{aligned}&= \frac{\pi}{2}\frac{1}{2}\int_0^{\pi}\left(\frac{x\sin^3(x)}{1+\cos^2(x)} + \frac{(\pi-x)\sin^3(x)}{1+\cos^2(x)}\right)\mathrm{d}x \\ &= \frac{\pi^2}{4}\int_0^{\pi}\frac{\sin^3(x)}{1+\cos^2(x)}\,\mathrm{d}x = \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^3(x)}{1+\cos^2(x)}\,\mathrm{d}x \quad \boxed{\int_0^{\pi} f(\sin\theta)\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} f(\sin\theta)\,\mathrm{d}\theta} \\ &= \frac{\pi^2}{2}\int_0^1 \frac{1-u^2}{1+u^2}\,\mathrm{d}u = \frac{\pi^2}{2}\int_0^1\left(-1+\frac{2}{1+u^2}\right)\mathrm{d}u \\ &= \frac{\pi^2}{2}\Big(2\tan^{-1}\big(1\big)-1\Big) = \frac{\pi^2}{4}(\pi-2)\end{aligned}$$

下面是一個特殊的積分技巧，叫做**組合積分法**。找出一個長相相近的積分式，通過解方程式來求積分。

例題 1.6

$$\int \frac{\mathrm{d}x}{(x+1)(x-1)}$$

解

設

$$I = \int \frac{\mathrm{d}x}{(x+1)(x-1)}, \quad J = \int \frac{x\,\mathrm{d}x}{(x+1)(x-1)}$$

於是有

$$\begin{cases} I + J = \int \frac{(x+1)\,\mathrm{d}x}{(x+1)(x-1)} = \int \frac{\mathrm{d}x}{x-1} = \ln|x-1| + C \\ J - I = \int \frac{(x-1)\,\mathrm{d}x}{(x+1)(x-1)} = \int \frac{\mathrm{d}x}{x+1} = \ln|x+1| + C \end{cases}$$

解方程式便可得

$$I = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

例題 1.7

$$\int \frac{\mathrm{d}x}{(1+x)(1+x^2)}$$

解

設

$$I = \int \frac{\mathrm{d}x}{(x+1)(x^2+1)}, \quad J = \int \frac{x^2\,\mathrm{d}x}{(x+1)(x^2+1)}$$

於是有

$$\begin{cases} I + J = \int \frac{(x^2+1)\,\mathrm{d}x}{(x+1)(x^2+1)} = \int \frac{\mathrm{d}x}{1+x} = \ln|1+x| + C \\ J - I = \int \frac{(x^2-1)\,\mathrm{d}x}{(x+1)(x^2+1)} = \int \frac{(x-1)\,\mathrm{d}x}{x^2+1} = \int \frac{x\,\mathrm{d}x}{x^2+1} - \int \frac{\mathrm{d}x}{x^2+1} = \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}(x) + C \end{cases}$$

解方程式便可得

$$I = \frac{1}{2}\left[\ln|x+1| - \frac{1}{2}\ln\left|x^2+1\right| + \tan^{-1}(x)\right] + C$$

例題 1.8

$$\int \frac{7\cos(x) - 3\sin(x)}{5\cos(x) + 2\sin(x)}\,\mathrm{d}x$$

解

設

$$I = \int \frac{\sin(x)}{5\cos(x) + 2\sin(x)}\,\mathrm{d}x, \quad J = \int \frac{\cos(x)}{5\cos(x) + 2\sin(x)}\,\mathrm{d}x$$

於是有

$$\begin{cases} 2I + 5J = \int \mathrm{d}x = x + C \\ 2J - 5I = \int \dfrac{2\cos(x) - 5\sin(x)}{5\cos(x) + 2\sin(x)}\,\mathrm{d}x = \ln\left|5\cos(x) + 2\sin(x)\right| + C \end{cases}$$

解方程式便可得

$$\begin{cases} I = \dfrac{1}{29}\left[2x - 5\ln\left|5\cos(x) + 2\sin(x)\right|\right] + C \\ J = \dfrac{1}{29}\left[5x + 2\ln\left|5\cos(x) + 2\sin(x)\right|\right] + C \end{cases}$$

則所求為

$$7J - 3I = x + \ln\left|5\cos(x) + 2\sin(x)\right| + c$$

例題 1.8